Algorithmic Aversion and Robo-Advisors

Alberto G Rossi

Cherry Blossom Financial Education Institute, 2023
Research Agenda on Robo-Advising

**Common Perception:**
Robo-advising = automated advice for portfolio allocation
Research Agenda on Robo-Advising

- **BUT** households’ decisions are more complex!

**Robo-Advising**: automated advice for ANY household choice

![The Balance Sheet View of Households](image)

(D’Acunto and Rossi, 2021)
Research Agenda on Robo-Advising

Robo-advising for Investment Decisions

- “Robo-advising,” D’Acunto & Rossi
- “The Promises and Pitfalls of Robo-advising,” D’Acunto, Prabhala & Rossi
- “Algorithmic Aversion: Theory and Evidence from Robo-advice,” Ramadorai et. al

Robo-advising/FinTech for Consumption, Saving, Debt & Lending

- “New Frontiers of Robo-Advising: Consumption, Saving, Debt, and Taxes,” D’Acunto and Rossi
- “Crowdsourcing Peer Information to Change Spending,” D’Acunto, Rossi & Weber
- “Goal Setting and Saving in the FinTech Era” Gargano & Rossi
- “Improving Households’ Debt Management with Robo-advising” D’Acunto, et. al
Algorithm Aversion: Theory and Evidence from Robo-advice

Fiona Greig    Tarun Ramadorai    Alberto G Rossi
Steve Utkus    Ansgar Walther
Motivation

- Algorithms are ubiquitous, potentially bringing huge benefits across household markets.

- **BUT** lack of trust in algorithms may limit the scalability of tech-enabled innovation

- Algorithm aversion:
  - Barrier to tech adoption, repeatedly confirmed in psych lab findings (Dietvorst et al., 2015).
  - Little consensus about psychological underpinnings (Burton et al., 2020).

- Are there *natural limits* to the algo-adoption arising from human preferences and beliefs?

**This paper:**

- Understanding the different components of algorithm aversion.

- Quantifying their relative importance using the important setting of hybrid robo-advice.
This Paper

- What drives algorithm aversion?
  - Theory: Preference- and belief-based aversion; Structural estimation: Investment robo-advice

1 Model: Dynamic Model of Automation Adoption
  - Robo-advisor with uncertain performance
  - Algo aversion: disutility, uncertainty, pessimism

2 Data: US Hybrid robo-advisor
  - Robo-portfolio plus quasi-random assignment to high/low-“type" human advisors.

3 Reduced-form: Causal impact of the human component
  - High-type human advice reduces base quit rate by 23%
  - Value of human advice especially evident in bad market conditions

4 Structural inference
  - Ongoing disutility (preference) more important than pessimism (belief).
  - Human advice reduces uncertainty by > 90%
General Model of Automation Adoption

NOTE:
We also provide a version of this general model for a Campbell-Viceira style portfolio choice problem featuring an investor with log utility of terminal wealth, who can sign-up for robo-advice, and faces disutility when interacting with an algorithm.
General Model of Automation Adoption

- Investors indexed by \( i = 1, \ldots, I \) can use automated service for up to \( T \) periods.
- Investors quit at (endogenously) chosen time \( \delta_i \), where \( \delta_i \in \{1, \ldots, T\} \).
- Quality of the service: \( \theta \in \mathbb{R} \).

- Investors randomly assigned to advisors/experts indexed by \( j = 1, \ldots, J \), who affect:
  1. clients’ utility from consuming the service
  2. clients’ beliefs at sign-up (\( t = 0 \)): \( \theta \sim N \left( m^j_0, \frac{1}{\tau^j_0} \right) \)
General Model of Automation Adoption

- Post-enrollment, $t \in \{1, \ldots, \delta_i\}$, learning from performance:

\[ y^i_t = \theta + u^i_t \quad \text{with} \quad u^i_t \sim N\left(0, \frac{1}{\tau_y}\right) \]

- At date $t$, beliefs about $\theta$ updated using Bayes’ rule using $\{y^i_1, \ldots, y^i_t\}$.

- Beliefs about $\theta$ are updated using the Kalman filter:

\[ \theta \sim N\left(m^{i,j}_t, \frac{1}{\tau_t^j}\right) \]

\[ m^{i,j}_t = m^{i,j}_{t-1} + \frac{\tau_y}{\tau_{t-1}^j + \tau_y} \left(y^i_t - m^{i,j}_{t-1}\right) \quad \& \quad \tau_t^j = \tau_0^j + t\tau_y \]

Note: depends on both $i$ and $j$
Bellman Equation for the investor’s problem:

\[
V^j_t(m) = \max \left\{ u^j(m) + \mathbb{E}^j \left[ V_{t+1}(m') \right| m \right\} , 0 \right\}
\]

- \( V^j_t(m) \): continuation value of client still enrolled at date \( t \) with expectations \( m^{i,j}_t = m \)
- \( u^j(m) \): expected flow of utility for a client who matches with expert \( j \)
- \( \mathbb{E}^j \left[ V_{t+1}(m') \right| m \] : client’s expected continuation value if enrolled until \( t + 1 \)
- Utility investor can obtain outside the service normalized to “0”
General Model of Automation Adoption

- For $t < T$, optimal to quit on date $t$, i.e. $\delta_i = t$, if and only if:

  $$u^i (m) + \hat{E}^j \left[ V_{t+1} (m') \big| m \right] \leq 0.$$ 

- For structural estimation, we add implementation error: $\xi^i \sim N \left( 0, \frac{1}{\tau_{\xi}} \right)$
  (generates cross-sectional variation in quit rates conditional on performance)

  $\Rightarrow$ For $t < T$, $\delta_i = t$ if and only if:

  $$u^i (m - \xi^i) + \hat{E}^j \left[ V_{t+1} (m') \big| m \right] \leq 0.$$
Structural Mapping
Structural Estimation

- Ultimate Goal: Estimate all parameters from simulated dynamics
- This draft: Analytical estimation equation for three-period model: $T = 2$
- Advantage: Clarifies parameter identification

Sign up at $t = 0$. $\delta_i = t$ if and only if:

$$u^i \left( m_{1}^{i,j} - \xi_{1}^{i} \right) + \hat{E}^{j} \left[ V_{2} \left( m_{2}^{i,j} \right) \right] m_{1}^{i,j} \leq 0$$

$$\equiv 0$$

$$\Rightarrow u^i \left( m_{1}^{i,j} - \xi_{1}^{i} \right) \leq 0$$

$$\Rightarrow u^i \left( m_{0}^{i,j} \frac{\tau_{j}}{\tau_{0} + \tau_{y}} + \frac{\tau_{y}}{\tau_{0} + \tau_{y}} y_{1}^{i} - \xi_{1}^{i} \right) \leq 0$$
Structural Estimation

Define \( \phi^j \equiv (u^j)^{-1}(0) \):

- Critical value for quality that makes investor \( i \) indifferent between quitting and continuing.
- Interpretable as the fixed cost/disutility of participation when matched with expert \( j \).

\[ \Rightarrow \text{Equivalent quitting condition:} \]

\[ \phi^j - \frac{\tau^j_0}{\tau^j_0 + \tau_y} m^i_0 - \frac{\tau_y}{\tau^j_0 + \tau_y} y^i_1 + \xi^i_1 \geq 0 \]

- Baseline quit rate \( \uparrow \) if
  - Disutility of participation increases: \( \phi^j \uparrow \)
  - Prior belief about robo-performance decreases: \( m^i_0 \downarrow \)

- Sensitivity \( \uparrow \) if
  - precision of the performance signal increases: \( \tau_y \uparrow \)
  - precision of the prior decreases: \( \tau^j_0 \downarrow \)
Structural Estimation

Empirical estimation of the model

1. Group advisors into high retention (H) and low retention (L) using historical performance
2. Estimate separately preferences and belief parameters for investors assigned to
   - high-retention advisors: \( \{ \phi^H, m^H_0, \tau^H_0 \} \)
   - low-retention advisors: \( \{ \phi^L, m^L_0, \tau^L_0 \} \)

For causal interpretation:

1. Assignment of client \( i \) to human expert \( j \) should be independent of \( \{ \xi^i_t, u^i_t \} \).
2. Make sure the advisor-type measure is not mechanically related to clients’ attrition
   \( \rightarrow \) use leave-one-out estimator throughout our analysis
Data
## Advised Investor Characteristics

### Panel A. Demographic Characteristics

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<tr>
<th>Characteristic</th>
<th>N</th>
<th>mean</th>
<th>sd</th>
<th>p25</th>
<th>p50</th>
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<td>Age</td>
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<td>12.1</td>
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<td>Male</td>
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<td>1.0</td>
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<td>Tenure</td>
<td>54,744</td>
<td>13.5</td>
<td>9.1</td>
<td>3.8</td>
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### Panel B. Portfolio-Related Characteristics

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### Panel C. Asset Allocation Characteristics

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<td>0.000</td>
<td>0.000</td>
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<td>PctETF</td>
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### Panel D. Characteristics of Mutual Funds Held

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<td>0.093</td>
<td>0.027</td>
<td>0.078</td>
<td>0.083</td>
<td>0.096</td>
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<td>0.120</td>
<td>0.190</td>
<td>0.280</td>
<td>0.337</td>
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Performance of Robo-advised and Self-directed Investors

- Robo assigns 5 glide paths based on objectives / horizon /demographics
  - Mostly 4 indexed mutual funds: VTSAX, VTIAx, VBTLX, VTabX
  - 70% cross-section of returns explained by investors’ age
- 5 Principal Components: 80% of the variation in equity share
- Clustering: 96% of investors assigned to 2 glide-paths
Measuring Advisor Type

Investors with Assets above $500K are assigned to an advisor

Revealed preference approach to measuring advisor type using a leave-one-out estimator

Retention rate of advisor $j$ and client $i$ is:

- Ratio of clients assigned to advisor $j$ that do not quit *excluding* from the computations:
  - Client $i$
  - The date $t$ at which client $i$ quits (takes care of cross-sectional correlation in attrition)

- Results are robust to:
  - Splitting dataset in half
  - Different specifications for controlling cross-sectional correlation in attrition
Clients’ Assignment to Advisors-I

- During onboarding, clients are asked for the characteristics
- The roboadvisor generates a financial plan

- To complete sign-up, investors have to meet with an advisor
- Investors give their availability
- A scheduling system tracks advisors’ availability

- All advisors have the same target number of clients
- Advisor Managers determine advisors’ onboarding rate

⇒ Assumption:
Investors’ assignment to Advisors is quasi-random, conditional on factors driving onboarding.
<table>
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<th>High Retention</th>
<th></th>
<th>Low Retention</th>
<th></th>
<th>Diff</th>
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<td>65.657</td>
<td>23,511</td>
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<td>(4.76)</td>
<td>49,824</td>
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<td>25,739</td>
<td>15.635</td>
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<td>0.969***</td>
<td>(3.03)</td>
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<td>25,739</td>
<td>993,861</td>
<td>24,085</td>
<td>47,107</td>
<td>(0.39)</td>
<td>49,824</td>
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<td>25,739</td>
<td>11.438</td>
<td>24,085</td>
<td>0.721</td>
<td>(1.69)</td>
<td>49,824</td>
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<tr>
<td>PctVGProducts</td>
<td>0.853</td>
<td>25,706</td>
<td>0.850</td>
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<td>(-0.81)</td>
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<td>(0.61)</td>
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<td>0.002</td>
<td>24,062</td>
<td>0.000**</td>
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<td>24,084</td>
<td>0.002</td>
<td>(0.14)</td>
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<td>MgtFee</td>
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<td>22,931</td>
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<td>(0.28)</td>
<td>46,808</td>
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<td>ExpRatio</td>
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<td>TurnRatio</td>
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<td>21,787</td>
<td>0.016**</td>
<td>(2.16)</td>
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<td>Ret. Pre-PAS</td>
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<td>22,040</td>
<td>0.045</td>
<td>20,884</td>
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<td>42,924</td>
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<td>22,040</td>
<td>-0.009</td>
<td>20,884</td>
<td>-0.002</td>
<td>(-1.64)</td>
<td>42,924</td>
</tr>
</tbody>
</table>

⇒ Investors assigned to low- and high-retention advisors are virtually indistinguishable.
Validating Quasi-Random Assignment-II

(a) All Advisors
(b) High Retention; Low Retention

⇒ High- and Low-retention advisors are assigned clients at the same rate
Empirical Estimates
Heterogeneity in Client Retention and Advisor Type

Advisors have different (scaled) client retention in the cross-section

But advisor fixed effect has no explanatory power for returns
Non Parametric Survival Estimates-I

- Compute advisors’ retention using leave-one-out estimator

- Split them into two groups:
  - Type 0: advisor with leave-one-out retention below median
  - Type 1: advisors with leave-one-out retention above median

- Take all investors signing up for PAS

- Estimate $\hat{S}(t) = \Pi_{i: t_i \leq t} \left( 1 - \frac{d_i}{n_i} \right)$ for each group
  where:
  - $t_i$: time when at least one investor quits
  - $d_i$: number of clients quitting robo-advice at time $t$
  - $n_i$: number of clients who have stayed with robo-advice
Cox Model: clients assigned to Type-1 human advisors have **25.4%** lower hazard
Effect of Human Advice Across Market Conditions-I

Regression Results:

\[ Dummy_{\text{quit},t} = \alpha + \beta I_{\{\text{MKT\_RET}_{t-1}<0\}} + \gamma I_{\{\text{Type1\_Advisor}_i=1\}} + \delta I_{\{\text{MKT\_RET}_{t-1}<0\}} \times I_{\{\text{Type1\_Advisor}_i=1\}} + \epsilon_{i,t}, \]

where

- \( Dummy_{\text{quit},t} \): 1 if investor \( i \) quits in month \( t \);
- \( I_{\{\text{MKT\_RET}_{t-1}<0\}} \): 1 if market returns are negative in month \( t-1 \)
- \( I_{\{\text{Type1\_Advisor}_i=1\}} \): 1 if investor \( i \) is assigned to a high-retention advisor
<table>
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<th>CRSP Return</th>
<th>Investor Return</th>
<th>CRSP Volatility</th>
<th>Investor Volatility</th>
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<td>Bad Market</td>
<td>0.136***</td>
<td>0.145***</td>
<td>0.125***</td>
<td>0.117***</td>
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<tr>
<td></td>
<td>(2.98)</td>
<td>(3.96)</td>
<td>(3.20)</td>
<td>(4.02)</td>
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<td>Type1_Advisor</td>
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<td>-0.078***</td>
<td>-0.051***</td>
<td>-0.056***</td>
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<tr>
<td></td>
<td>(-4.74)</td>
<td>(-4.03)</td>
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<td>Interaction</td>
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<td>-0.123***</td>
<td>-0.102***</td>
<td>-0.091***</td>
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<tr>
<td></td>
<td>(-3.77)</td>
<td>(-3.90)</td>
<td>(-3.21)</td>
<td>(-3.61)</td>
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<tr>
<td>Constant</td>
<td>0.369***</td>
<td>0.359***</td>
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<tr>
<td></td>
<td>(17.70)</td>
<td>(16.23)</td>
<td>(17.78)</td>
<td>(17.87)</td>
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Clustering: Date&User

R-square: 0.00011 0.00013 0.00014 0.00013
N: 938,314 938,314 938,314 938,314
Effect of Human Advice Across Market Conditions-III

Using Column (1) coefficients:

(a) Good Market Conditions
- Low Retention advisors lose more clients in bad markets
- High Retention advisors perform similarly across market conditions

(b) Poor Market Conditions
Structural Mapping—Empirics-I

- Baseline quit rates:
  \[
  \phi^L - \frac{\tau^L_0}{\tau^L_0 + \tau_y} m^L_0 = 0.369 \times 12 = 4.43\% \\
  \phi^H - \frac{\tau^H_0}{\tau^H_0 + \tau_y} m^H_0 = (0.369 - 0.086) \times 12 = 3.40\% 
  \]

  \[ \Rightarrow \text{High-type advisor reduces baseline quit rates by } 1 - 3.4/4.43 = 23.25\% \]

- Sensitivity to performance:
  \[
  \frac{\tau_y}{\tau^L_0 + \tau_y} = 0.136 \times 12 = 1.63\% \\
  \frac{\tau_y}{\tau^H_0 + \tau_y} = (0.136 - 0.123) \times 12 = 0.16\% 
  \]

  \[ \Rightarrow \text{High-type advisor reduces sensitivity to performance by } 1 - 0.16/1.63 = 90.18\% \]
**Structural Mapping—Empirics-II**

- Does sensitivity vary with tenure?

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<th>Long Tenure</th>
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<td>0.044 (1.36)</td>
</tr>
<tr>
<td><strong>Type1_Advisor</strong></td>
<td>-0.057*** (-3.29)</td>
<td>-0.089*** (-3.62)</td>
</tr>
<tr>
<td><strong>Interaction</strong></td>
<td>-0.158*** (-3.55)</td>
<td>-0.102** (-2.19)</td>
</tr>
<tr>
<td><strong>Constant</strong></td>
<td>0.302*** (23.77)</td>
<td>0.417*** (16.95)</td>
</tr>
</tbody>
</table>

For long tenure investors:

\[
\frac{\tau_0^L}{\tau_0^L + \tau_y} m_0^L \simeq \hat{\phi}^L = 0.417 \times 12 = 5.00\%
\]

\[
\frac{\tau_0^H}{\tau_0^H + \tau_y} m_0^H \simeq \hat{\phi}^H = (0.417 - 0.089) \times 12 = 3.94\%
\]

\[
\Rightarrow \text{High-type advisor reduces baseline fixed cost/disutility by } 1 - \frac{3.94}{5} = 21.2\%
\]
Conclusions

- We present a model of algorithm aversion featuring:
  - Learning about algorithm’s ability
  - Ongoing disutility from using the algorithmic solution

- We map the model to the real world using
  - Data from a hybrid robo-advisor (PAS)
  - Quasi-random assignment of clients to advisors

- Main findings
  - Significant algo aversion reduced by human advisors
  - High-type advisors retain more clients in turbulent times
  - Experienced clients react less to market turbulence
  - Uncertainty and disutility channels of algorithm aversion are structurally most important
Appendix Slides
Model Specialized to Portfolio Choice and Robo-advice-I

- Investor can allocate fraction \( \alpha_t \geq 0 \) to a robo-advisor. \( 1 - \alpha_t \) invested outside portfolio.
- Client’s utility is \( w_T = \ln(W_T) \), where \( W_T \) is the final wealth.
- When robo-advised, client suffers a fixed cost \( f^j \): depends on the identity of human advisor.
- Log return on the outside portfolio is deterministic and given by \( \bar{r} \).
- Robo-advisor generates stochastic log returns given by \( r_{t+1}^i = \bar{r} + \theta + u_{t+1}^i \).
- Investor’s beliefs about \( \theta \), as a function of her human advisor \( j \), are as in the general model.
- Investor’s log wealth, evolves according to the following approximate law of motion:

\[
 w_{t+1}^i - w_t \simeq \bar{r} + \alpha_t y_{t+1}^i + \frac{1}{2} \sigma^2 \alpha_t (1 - \alpha_t), \quad \text{where} \quad \sigma^2 = 1/\tau_y 
\]  

\text{Proof} \quad (1)
Conjecture that the investor’s continuation value, if still enrolled at date $t < T$, is:

$$F_t (w, m) = w + (T - t) \bar{r} + V_t (m).$$

At date $t$, two options:

- Quit $\Rightarrow$ final utility takes the deterministic value $w + (T - t) \bar{r}$.

- Stay robo-advised with an optimally chosen portfolio weight $\alpha_t$.

Investor’s Bellman Equation:

$$F_t (w, m) = \max \left\{ w + (T - t) \bar{r}, -f^j + \max_{\alpha \geq 0} \mathbb{E}^j \left[ F_{t+1} (w', m') \mid w, m, \alpha_t = \alpha \right] \right\}$$

(2)
Substitute conjecture and law of motion of wealth to write the last term of Equation (??) as:

$$
\hat{E}^j \left[ F_{t+1} \left( w', m' \right) \mid w, m, \alpha_t = \alpha \right] = w' + (T - (t + 1)) \bar{r} + \hat{E}^j \left[ V_t \left( m' \right) \mid m \right]
$$

$$
= w + (T - t) \bar{r} + \alpha m + \frac{1}{2} \sigma^2 \alpha \left( 1 - \alpha \right) + \hat{E}^j \left[ V_t \left( m' \right) \mid m \right]
$$

(3)

where we use that: $m = \hat{E}^j \left[ y' \mid m \right]$ by definition

Inner maximization in Equation (??) is solved by optimal portfolio weight

$$
\hat{\alpha} = \max \left\{ \frac{m + \frac{1}{2} \sigma^2}{\sigma^2}, 0 \right\},
$$
Model Specialized to Portfolio Choice and Robo-advice-IV

The value of (??) is:

\[ E[F_{t+1} (w', m') | w, m, \alpha_t = \hat{\alpha}] = w + (T - t) \bar{r} + \left( m + \frac{1}{2} \sigma^2 \right) \hat{\alpha} - \frac{1}{2} \sigma^2 \hat{\alpha}^2 + \hat{E}^j [V_{t+1} (m') | m] \]

\[ = w + (T - t) \bar{r} + \hat{E}^j [V_{t+1} (m') | m] + \frac{1}{2} [SR (m)]^2, \]

where

\[ SR (m) = \begin{cases} 0, & m + \frac{1}{2} \sigma^2 < 0, \\ \frac{m + \frac{1}{2} \sigma^2}{\sigma}, & \text{otherwise}. \end{cases} \]

Substitute this into (??), together with the conjecture solution, to obtain:

\[ V_t (m) = \max \left\{ -f^j + \frac{1}{2} [SR (m)]^2 + \hat{E}^j [V_{t+1} (m') | m], 0 \right\}, \]

which maps into our general model when the investor’s utility function is:

\[ u^j (m) = -f^j + \frac{1}{2} [SR (m)]^2. \]
Derivation of Intertemporal Budget Constraint-I

Here we provide the derivation of the law of motion of wealth, i.e., Equation (??).

- Consider continuous-time model where per-unit value $P_t$ of robo portfolio follows:

$$\frac{dP_t}{P_t} = \left( \bar{r} + \theta + \frac{1}{2} \sigma^2 \right) dt + \sigma dZ_t,$$

where $Z_t$ is a standard Brownian Motion

- Discrete-time representation:

$$r_{t+1} \equiv \log \left( \frac{P_{t+1}}{P_t} \right) = \bar{r} + \theta + \sigma u_{t+1}, \text{ where } u_{t+1} = Z_{t+1} - Z_t$$

- Outside portfolios evolves according to

$$\frac{dB_t}{B_t} = \bar{r} dt$$
Derivation of Intertemporal Budget Constraint-II

- Investor’s wealth evolves according to:

\[
\frac{dW_t}{W_t} = \alpha_t \frac{dP_t}{P_t} + (1 - \alpha_t) \frac{dB_t}{B_t}
\]

\[
= \alpha_t \left[ \left( \bar{r} + \theta + \frac{1}{2} \sigma^2 \right) dt + \sigma dZ_t \right] + (1 - \alpha_t) \bar{r} dt
\]

\[
= \left[ \alpha_t \left( \bar{r} + \theta + \frac{1}{2} \sigma^2 \right) + (1 - \alpha_t) \bar{r} \right] W_t dt + \alpha_t \sigma W_t dZ_t
\]

- Converting to log returns, and applying Ito’s lemma to \( f(W) = \log W \), we obtain:

\[
d \log W_t = df(W_t) = f'(W_t) dW_t + \frac{1}{2} f''(W_t) (dW_t)^2 dt
\]

\[
= \left[ \alpha_t \left( \bar{r} + \theta + \frac{1}{2} \sigma^2 \right) + (1 - \alpha_t) \bar{r} \right] dt + \alpha_t \sigma dZ_t - \frac{1}{2} \left( \alpha_t \sigma \right)^2 dt
\]

\[
= \left[ \alpha_t \left( \bar{r} + \theta \right) + (1 - \alpha_t) \bar{r} + \frac{1}{2} \sigma^2 \alpha_t (1 - \alpha_t) \right] dt + \alpha_t \sigma dZ_t
\]
Derivation of Intertemporal Budget Constraint-III

- For discrete time approximation, set $dt = 1$ to get:

$$\log W^i_{t+1} - \log W_t = \alpha_t (\bar{r} + \theta) + (1 - \alpha_t) \bar{r} + \frac{1}{2} \sigma^2 \alpha_t (1 - \alpha_t) + \alpha_t \sigma u_{t+1}$$

$$= \alpha_t (\bar{r} + \theta + \sigma u_{t+1}) + (1 - \alpha_t) \bar{r} + \frac{1}{2} \sigma^2 \alpha_t (1 - \alpha_t)$$

$$= \alpha_t r_{t+1} + (1 - \alpha_t) \bar{r} + \frac{1}{2} \sigma^2 \alpha_t (1 - \alpha_t)$$

$$= \bar{r} + \alpha_t (r_{t+1} - \bar{r}) + \frac{1}{2} \sigma^2 \alpha_t (1 - \alpha_t)$$

$$= \bar{r} + \alpha_t y_{t+1} + \frac{1}{2} \sigma^2 \alpha_t (1 - \alpha_t) ,$$

which establishes Equation (??), where we have again used $u_{t+1} = Z_{t+1} - Z_t$. 

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