GFLEC
GLOBAL FINANCIAL LITERACY EXCELLENCE CENTER

Risk, Leverage, and Retirement Planning

Lecture 11

Leverage

In this lecture, we discuss **leverage**. Leverage is the use of debt to increase returns, and is best introduced using an example.

Ex. A young entrepreneur sees a business opportunity that requires an investment of $10,000.

Although the entrepreneur only has $2,500 to invest, she is able to convince her parents to lend her the remaining $7,500 (without interest). The project yields a 10% return on investment.

Compare the return the entrepreneur receives on her $2,500 investment to the return she would have received if she had funded the entire $10,000 investment.

Leverage

Ans.

Because the project returned 10%, the final value of the invested assets is:

$$A_F = \$10{,}000 * 1.10 = \$11{,}000$$

If the entrepreneur had funded the entire $10,000 investment, her return would (quite obviously) have been 10%:

$$R = \frac{\$11{,}000}{\$10{,}000} - 1 = 10\%$$

However, when debt is introduced, the return must be calculated on the investor's **equity**, or her original $2,500 investment. The entrepreneur's original equity is $2,500, while her debt is $7,500. This must, and does, satisfy the accounting identity:

$$A_0 = \$10{,}000 = \$2{,}500 + \$7{,}500 = E_0 + D_0$$

Leverage

Ans. (continued)

When the value of the invested assets increases by 10% to $11,000, the investor's final equity can be found by subtracting her final debt. Because she pays no interest on her loan, her debt remains at $7,500, and her final equity is:

$$E_F = A_F - D_F = \$11{,}000 - \$7{,}500 = \$3{,}500$$

In other words, after her investment grows to $11,000 and she repays the $7,500 she owes her parents, she has $3,500. Therefore, her return is:

$$R = \frac{\$3{,}500}{\$2{,}500} - 1 = 40\%$$

This rate of return is **four times greater** than the rate of return she would have received if she had financed the entire project!

Leverage ratio

An investment is said to be **leveraged** if it is partially financed by debt, and one effect of leverage is that it **magnifies returns**.

Leverage is sometimes measured by the **leverage ratio**, which is the total investment divided by the amount the investor contributes (the investor's equity, or capital):

$$L = \frac{A}{E}$$

Where *L* is the leverage ratio, *A* is the amount of total invested assets, and *E* is the amount of equity invested.

Leverage ratio

When an investment is leveraged, it can be shown that the return on the invested equity is approximately equal to:

$$R \approx Lr$$

Where *R* is the return on equity, *L* is the leverage ratio, and *r* is the return on assets.

Ex. The young entrepreneur in last example invested \$2,500 of her own equity in \$10,000 of assets, making her leverage ratio four:

$$L = \frac{A}{E} = \frac{\$10{,}000}{\$2{,}500} = 4$$

The return on the assets was 10%. And, as we saw, the return on her equity was 40%. This satisfies the formula $R = Lr = 4 * 10\% = 40\%$.

Leverage and Risk

Leverage and risk

A general rule of finance is that increased return cannot be had without increased risk, and a levered return is no exception: **leverage magnifies profits, but it also magnifies losses**.

- The relationship between leverage, return on assets, and return on equity, $R = Lr$, holds even if the return on assets is negative.
- In the example of the young entrepreneur, if her project returns 30% half the time and *loses* 10% the other half of the time, it will return 10% on average. Her leveraged return will be 40%, on average.
- But since her business has a chance of *losing* 10%, she will have a chance of losing *40%* of her equity!

Leverage and risk

Ex.

Consider again the entrepreneur above who is able to finance a $10,000 project with $2,500 in her own equity and the remaining $7,500 with an interest-free loan from her parents.

In this case, however, analyze the project as a risky investment that returns 30% half the time but loses 10% the other half of the time.

What is the return for the entrepreneur when the project yields 30%? What is her return when the project loses 10%? What is her expected return?

Compare the risk of her return to what it would be if she was able to finance the entire $10,000 without debt.

Leverage and risk

Ans. When the project returns 30%, the entrepreneur's return is:

$$A_F = \$10{,}000 * 1.13 = \$13{,}000$$

$$E_F = A_F - D_F = \$13{,}000 - \$7{,}500 = \$5{,}500$$

$$R = \frac{E_F}{E_0} - 1 = \frac{\$5{,}500}{\$2{,}500} - 1 = 120\%$$

And when the project *loses* 10%, her return is:

$$A_F = \$10{,}000 * (1 - 0.10) = \$9{,}000$$

$$E_F = A_F - D_F = \$9{,}000 - \$7{,}500 = \$1{,}500$$

$$R = \frac{E_F}{E_0} - 1 = \frac{\$1{,}500}{\$2{,}500} - 1 = -40\%$$

Leverage and risk

Ans. (continued)

Note again, that in both cases the return on the entrepreneur's equity is simply the projects return multiplied by the leverage ratio:

$$L = \frac{A}{E} = \frac{\$10{,}000}{\$2{,}500} = 4$$

$$r = 30\% \rightarrow R = Lr = 4 * 30\% = 120\%$$

$$r = -10\% \rightarrow R = Lr = 4 * (-10\%) = -40\%$$

And her expected return is:

$$E[R] = 0.5 * 120\% + 0.5 * (-10\%) = 40\%$$

It should not be surprising that this is four times the expected return on the project (10%).

Leverage and risk

Ans. (continued)

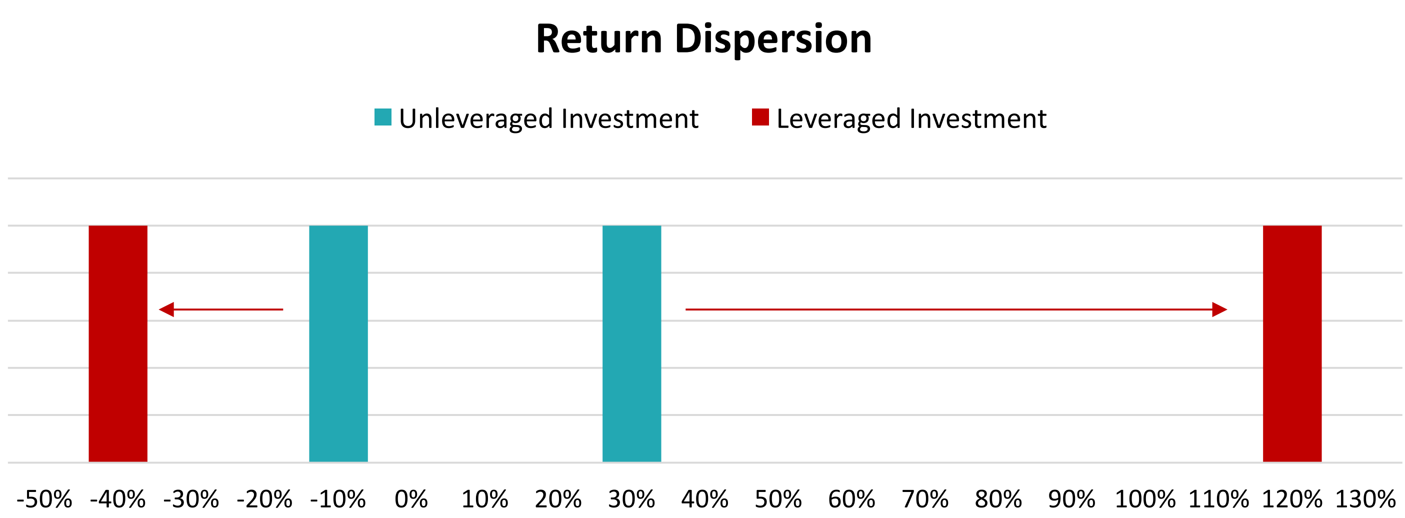
However, this increase in expected return is not free. As usual, this greater return is accompanied by greater risk.

For the project in general, the lowest possible return is 10%. But when the investment is leveraged, the investor faces a possible loss of 40%.

It can also be shown that the standard deviation of the return on the project is 20% and that the standard deviation of the leveraged investment is four times greater, at 80%!

Leverage and risk

While leveraging an investment increases the expected return, it also increases the dispersion of the returns, and hence, the risk.



Interest and Leverage

Leverage and interest on debt

A more explicit cost of leverage is the interest rate charged on the debt.

- The interest expense associated with debt will reduce an investor's leveraged returns to less than what they otherwise would be.
- Leverage magnifies not only the return on assets, but also the interest expense.
- If the interest expense on the debt exceeds the return on assets, leverage reduces the return to lower than what it would be without debt.

Leverage and interest on debt

It can be shown that the return on an investor's equity, after considering interest on debt, is governed by the formula:

$$R = Lr - \frac{D_0}{E_0} i = Lr - (L - 1)i$$

Where i is the interest rate on the debt, D_0 is the beginning debt level and E_0 is the amount of equity originally contributed by the investor.

The term $\frac{D_0}{E_0}$ is known as the **debt-to-equity ratio**.

Leverage and interest on debt

Both forms of the equation provide insight to the relationship between debt and returns. The first equation is:

$$R = r + \frac{D_0}{E_0}(r - i)$$

This tells us that the return on equity is the return on the investment, plus the excess return over the interest rate multiplied by the debt-to-equity ratio.

- If the return on the investment exceeds the interest rate on the debt, leverage increases the return.
- If the return falls below the interest rate on the debt, leverage decreases the return, possibly from positive to negative.
- In both cases, the increase or decrease is magnified by the amount of leverage.

Leverage and interest on debt

The second equation is:

$$R = Lr - \frac{D_0}{E_0} i$$

This tells us that the return on the investment is the return times the leverage ratio, minus the interest rate multiplied by the debt-to-equity ratio.

- When the interest rate on the debt is zero, the return is simply multiplied by the leverage ratio, as we saw earlier.
- Leverage not only increases the return, but also the cost of interest relative to the investment; the higher the debt-to-equity ratio, the more a given interest rate will detract from the overall return on equity.

Leverage and interest on debt

Ex 1. Apply the formula derived above to calculate an entrepreneur's return when her project is leveraged 3:1 (with a $10,000 investment financed with $2,500 of equity and $7,500 of debt), the project returns 10%, and she pays a 4% interest rate on her debt.

Ans. To calculate the return on the entrepreneur's original $2,500 investment, apply the formula:

$$R = r + \frac{D_0}{E_0}(r - i) = 10\% + \frac{3}{1} * (10\% - 4\%) = 28\%$$

This is considerably less than the 40% return the entrepreneur would receive if no interest was charged.

In fact, although the interest rate is only 4%, the return was lowered by 12%. In general, the effect of the interest rate is larger than the interest rate itself, because it is magnified by the debt-to-equity ratio.

Leverage and interest on debt

Ex 2. Now re-examine the project when there is a 4% interest rate on debt and risk is reintroduced. Again assume a $10,000 investment funded by $2,500 in equity and $7,500 in debt. Assume the project returns 30% half the time but loses 10% the other half of the time. Calculate the expected return of the project.

Ans. Using the formula, the return can be calculated for both scenarios:

$$r = 30\% \rightarrow R = 30\% + 3 * (30\% - 4\%) = 108\%$$
$$r = -10\% \rightarrow R = -10\% + 3 * (-10\% - 4\%) = -52\%$$

In both scenarios, the return is reduced by 12% from what it would be without any interest expense. If it weren't bad enough that a 10% loss would be magnified to a 40% loss without interest, the interest expense further increases the loss to 52%.

Leverage and interest on debt

Ans. (continued)

In sum the interest decreased the expected return from 40% to 28%. In general:

$$E[R] = L * E[r] - \frac{D_0}{E_0} * i$$

The expected return on equity is not simply magnified by the leverage, but also may be significantly reduced by the interest expense.

It can be shown, however, that the risk continues to be magnified by the leverage ratio. In this example, while the expected return was only increased by a factor of 2.8 after interest was considered, the standard deviation of the return continues to increase by a factor of 4, from 20% to 80%.

Leverage and Bankruptcy

Leverage and the risk of ruin

Because leverage magnifies losses, it can increase the risk of bankruptcy. A business that borrows too heavily may be put out of business by one unprofitable year!

- A business is **insolvent**, and will have trouble paying its obligations, if it losses 100% of its equity.
- An unleveraged business would need to incur a 100% loss in its business for the to occur.
- A business leveraged 2:1 that borrows at 5% would need only to incur a loss of 48% because:

$$R = Lr - (L-1)i = 2 * (-48\%) - 1 * 5\% = -101\%$$

- A firm that borrows 90% of its capital (10:1 leverage), will be insolvent if its business incurs a loss of only 6%!

Leverage and the risk of ruin

Ex. Consider again the case of an entrepreneur who undertakes a \$10,000 project financed by \$2,500 in equity. The remainder is financed with \$7,500 in debt at an interest rate of 4%. What loss is sufficient to drive her into insolvency?

Ans. The entrepreneur will be insolvent when the return on her equity is negative 100% or less: $R \leq -100\%$

By substituting the leverage formula, we can solve for the return on assets that satisfies this inequality:

$$Lr - \frac{D_0}{E_0} * i = 4r - 3 * 4\% \leq -100\%$$

$$r \leq -22\%$$

A loss of only 22% is sufficient to make the business insolvent!

Insolvency loss rate

In general, the formula to compute the **insolvency loss rate**, or the loss rate on assets sufficient to push a leveraged business into insolvency, can be found as:

$$R = Lr - \frac{D_0}{E_0} * i = -100\%$$

$$\rightarrow l \equiv -r = \frac{100\%}{L} - \frac{D_0}{A_0} * i$$

This suggests that the insolvency loss rate, *l*, is first inversely proportional to the leverage ratio. **Doubling leverage halves the loss rate necessary to cause insolvency**. Then, because debt must be financed at a fixed rate regardless of the return on assets, **the insolvency loss rate is further lowered by the interest payment on debt**.

Leverage and insolvency

The following table lists the rate of loss on assets required to drive a business into insolvency for different leverage ratios and costs of debt:

LEVERAGE AND THE INSOLVENCY LOSS RATE							
	Interest Rate						
Leverage	0%	1%	2%	4%	5%	10%	15%
1	100%	100%	100%	100%	100%	100%	100%
2	50%	50%	49%	48%	48%	45%	43%
3	33%	33%	32%	31%	30%	27%	23%
4	25%	24%	24%	22%	21%	18%	14%
5	20%	19%	18%	17%	16%	12%	8%
10	10%	9%	8%	6%	6%	1%	-4%

The higher the interest rate on the debt, the less the rate of loss required to bankrupt a business!

Costs of leveraged investing

While leverage may be tempting because of its potential to increase the return on an investment, the costs are sobering when properly understood.

- Leverage has as much potential to increase losses as it does returns (leverage increases **financial risk**).
- When the cost of financing the debt is considered, leverage has an asymmetric effect on returns. It effectively magnifies losses more than gains. Consequently, it increases risk by *more* than it increases expected returns.
- Leverage increases the possibility that a business's capital will be entirely wiped out and that an entrepreneur will face the worst fate of a business owner: bankruptcy. There is little room for error in a highly leveraged business.

Real estate and leverage

Real estate purchases are usually highly leveraged by a mortgages. As an investment, this increases returns but makes real estate very risky.

- Financing a home with a mortgage requiring a 20% down payment implies that the borrower is leveraged 5:1.
- Some borrowers may be able to take out additional financing (with **piggyback loans**) and ultimately pay a down payment of only 5%. This is 20:1 leverage!
- If the house price rises, their wealth increases quickly and they may sell the house at a profit.
- But if the house price falls, they quickly lose **equity** in their home and may find that they are **underwater** – that is they may owe more on their mortgage than their house is worth!

Life-Cycle Investing

Financial crises and investment horizon

Earlier, we considered the effect of the timing of a stock market crash for retirement savings. We briefly review the results:

- We consider the effect over a **35-year investment horizon**.
- The investor contributes **$10,000 per year**.
- The investor realizes a return of **13%** in all but one year.
- In that year, the stock market crashes and the investor receives a return of **negative 47%**. (The frequency of crashes is impossible to predict, but such returns might be witnessed once every 50-100 years, based on historical experience).

Financial crises and investment horizon

Under these conditions it can be shown that:

Crises at retirement:

Ending wealth: $2,897,408
Average return: 9.9%

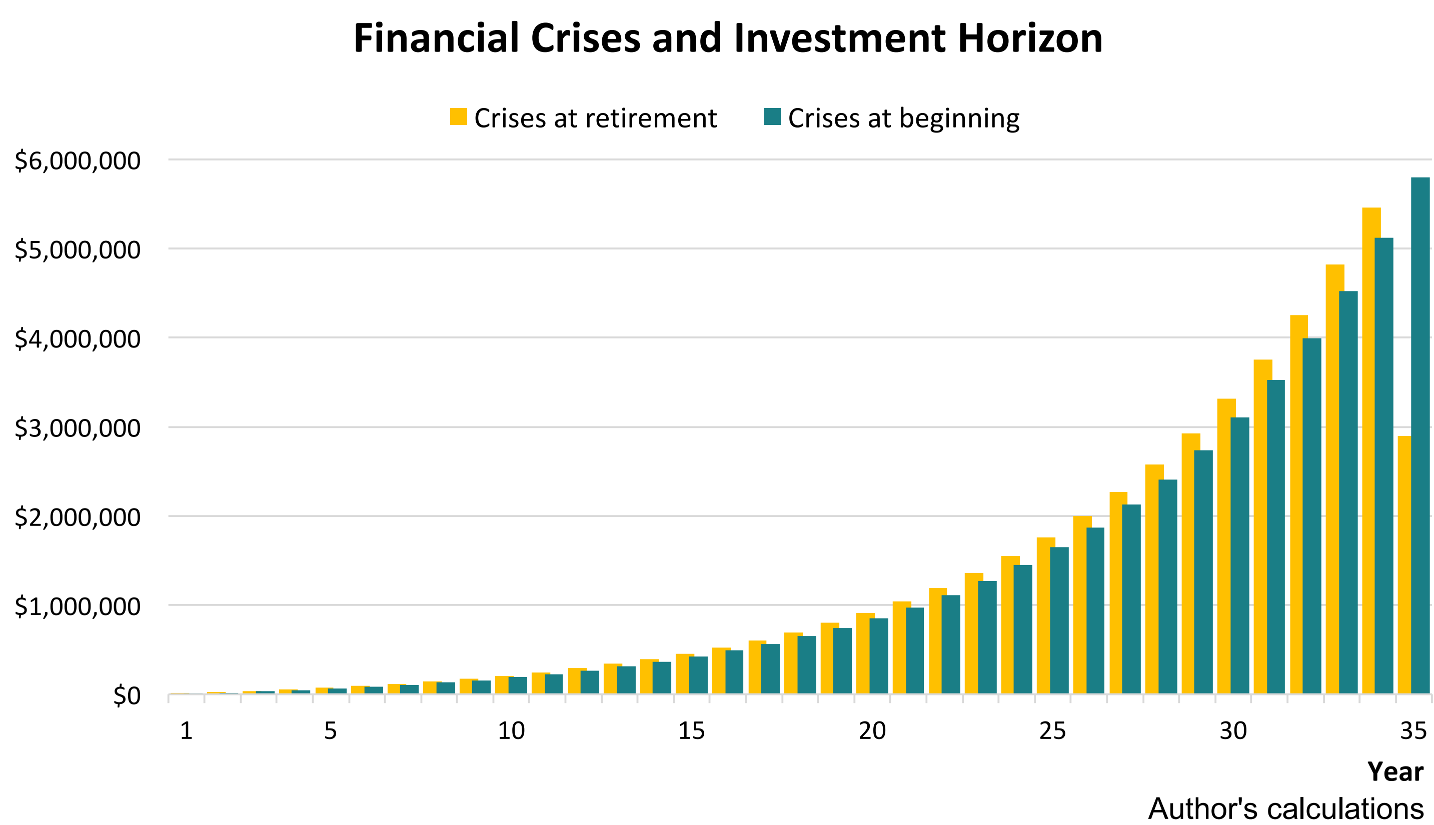
Crises at beginning of investment horizon:

Ending wealth: $5,794,828
Average return: 12.7%

The lesson from this result is clear: **a stock market crash at retirement has a devastating effect on final wealth.**

Financial crises and investment horizon

The following graph shows the growth of wealth under the two scenarios over time:



(Note the collapse in wealth when the crises occurs at retirement)

Implications

This example has **real-world implications**:

- If a crises occurs at the end of an investor's investment horizon, the investor's **entire accumulated wealth is affected**. But if the crises occurs at the beginning, only the investor's initial contributions are affected.
- Thus, when the investment horizon becomes short as an investor approaches retirement, the short-term volatility of the stock market threatens the investor's entire accumulated wealth.
- However, because the long-run expected return of the market is high and the volatility is averaged over long periods, it is sensible to invest in stocks when the horizon is long.
- Therefore, an investor would be well-advised to invest in stocks when young, but to transition out from stocks and into less risky assets, such as Treasuries, as retirement approaches.

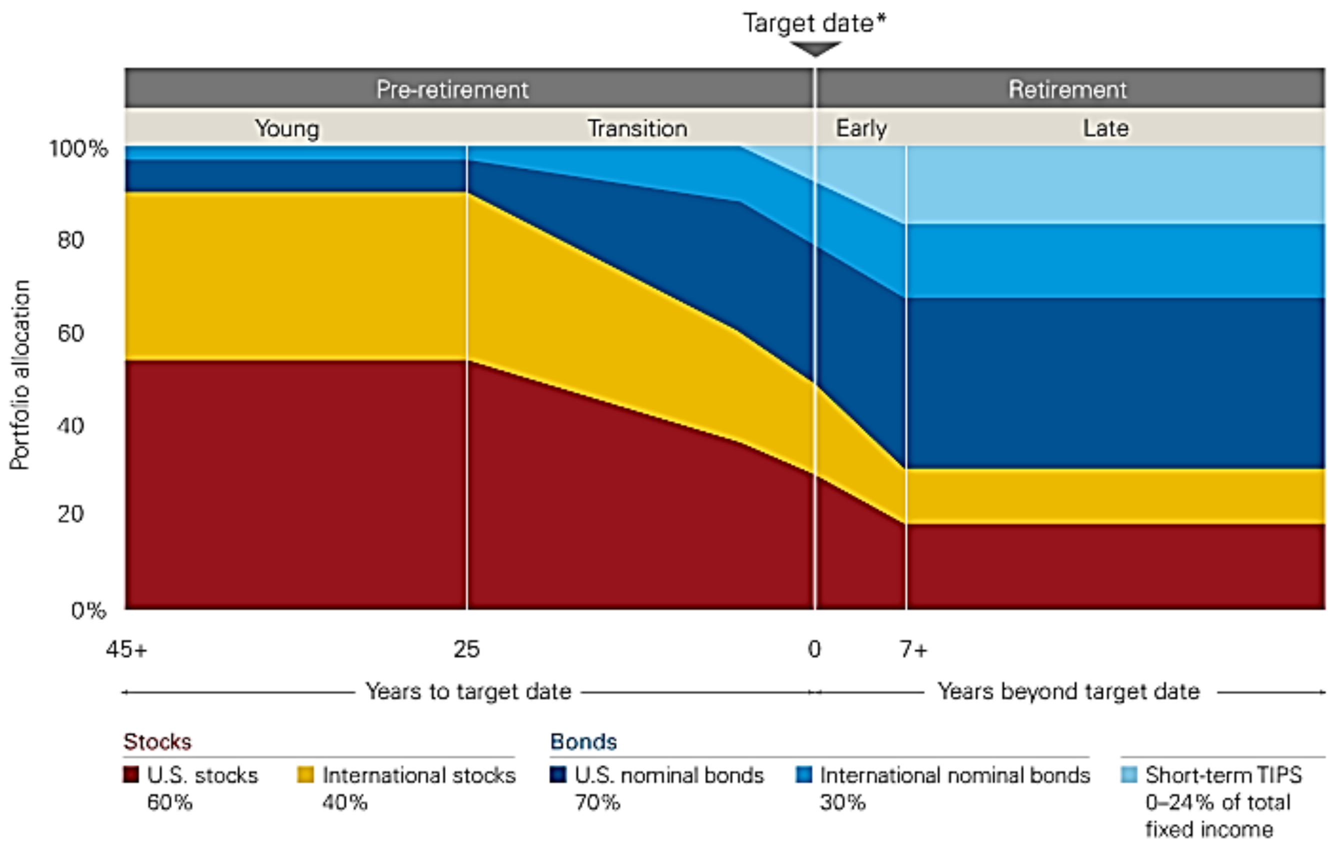
Life-cycle retirement investing

Investing in higher-yielding but riskier assets when young and transitioning into more conservative assets as retirement approaches is known as **life-cycle** investing.

- Riskier assets, such as stock, have offered and should continue to offer a return premium over long horizons.
- **Young investors with a long investment horizon are better positioned not only to take advantage of this long-term premium, but also to withstand the risks**; because they still have a large amount of future earnings, and less accumulated wealth in savings, they can withstand negative shocks.
- Investors are **less able to recover from shocks as they approach and enter retirement** and so it's important that they preserve their wealth.
- This logic suggests investors should invest aggressively when young but transition to more conservative assets as retirement approaches.

Target-date retirement funds

Some mutual fund companies offer funds that automatically rebalance and transition from aggressive to conservative as the investor approaches retirement:

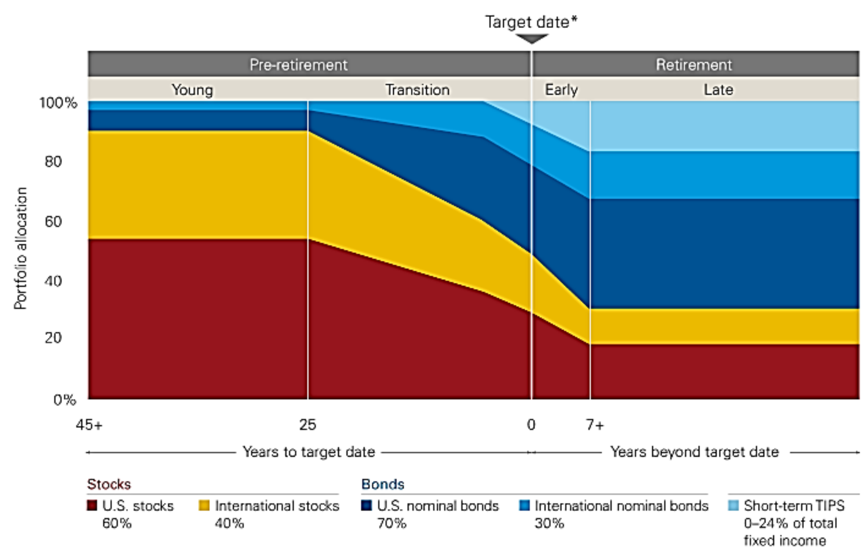


Source: Vanguard.

*Target date is the year stated in the fund name and assumes retirement at age 65.

Target-date retirement funds

- The asset allocation for Vanguard’s Target-Date Funds automatically adjust depending on how far away the investor’s selected target date is.
- Initially, the fund is 90% stock and 10% bonds.
- It then transitions to 40-60% stock through early retirement.
- (Note also the diversification: there is always some stock and some bonds, and a combination of domestic and international stock.)



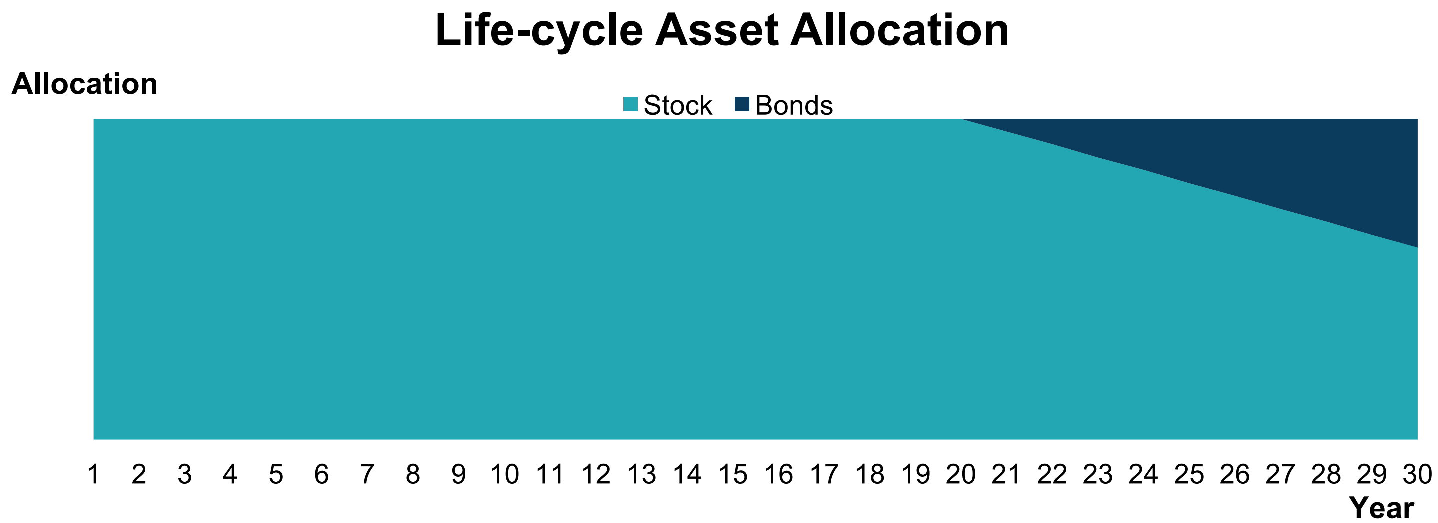
Source: Vanguard.

*Target date is the year stated in the fund name and assumes retirement at age 65.

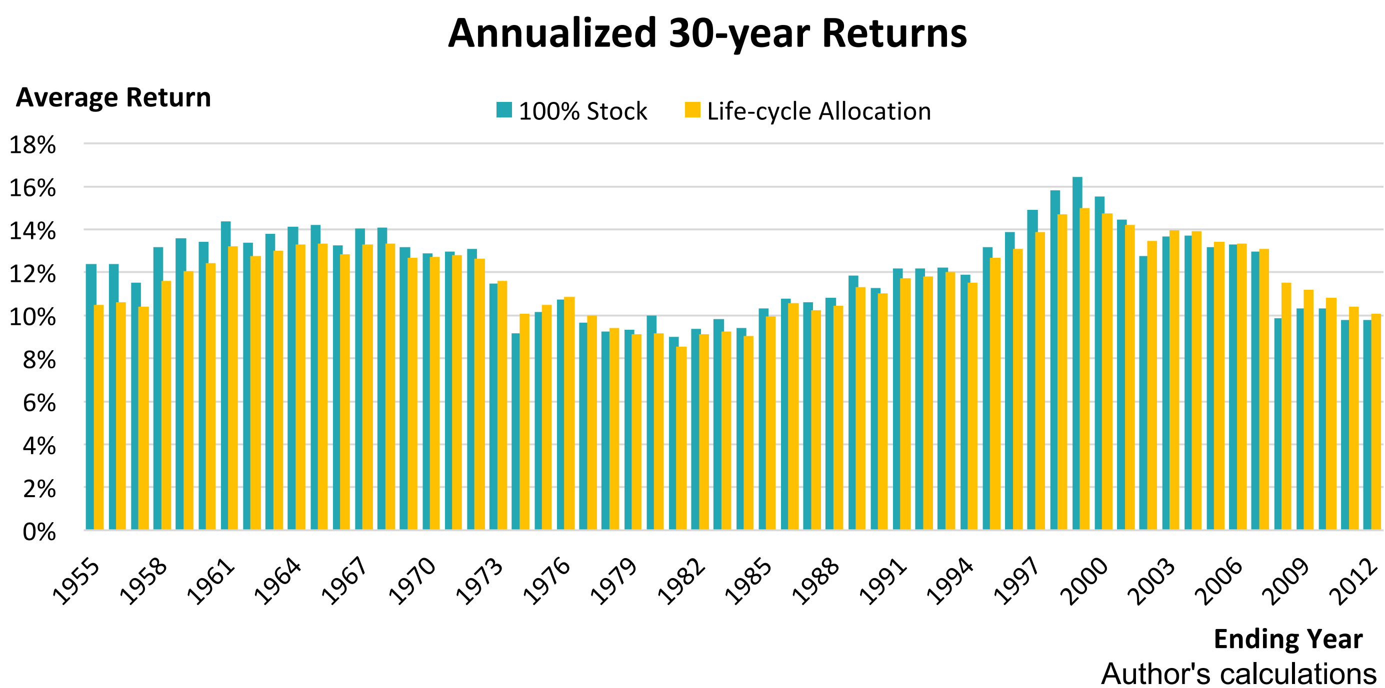
Example life-cycle portfolio

To see the effects of life-cycle investing, we now consider a basic simulation:

- An investor contributes $10,000 into his retirement account each year for the next 30 years.
- For the first 20 years, his portfolio is 100% stock.
- Over the final 10 years, the portfolio transitions from 100% stock to 60% stock and 40% bonds.
- We compare the ending wealth realized from this strategy for different investment periods (from 1926-1955 to 1983-2012).

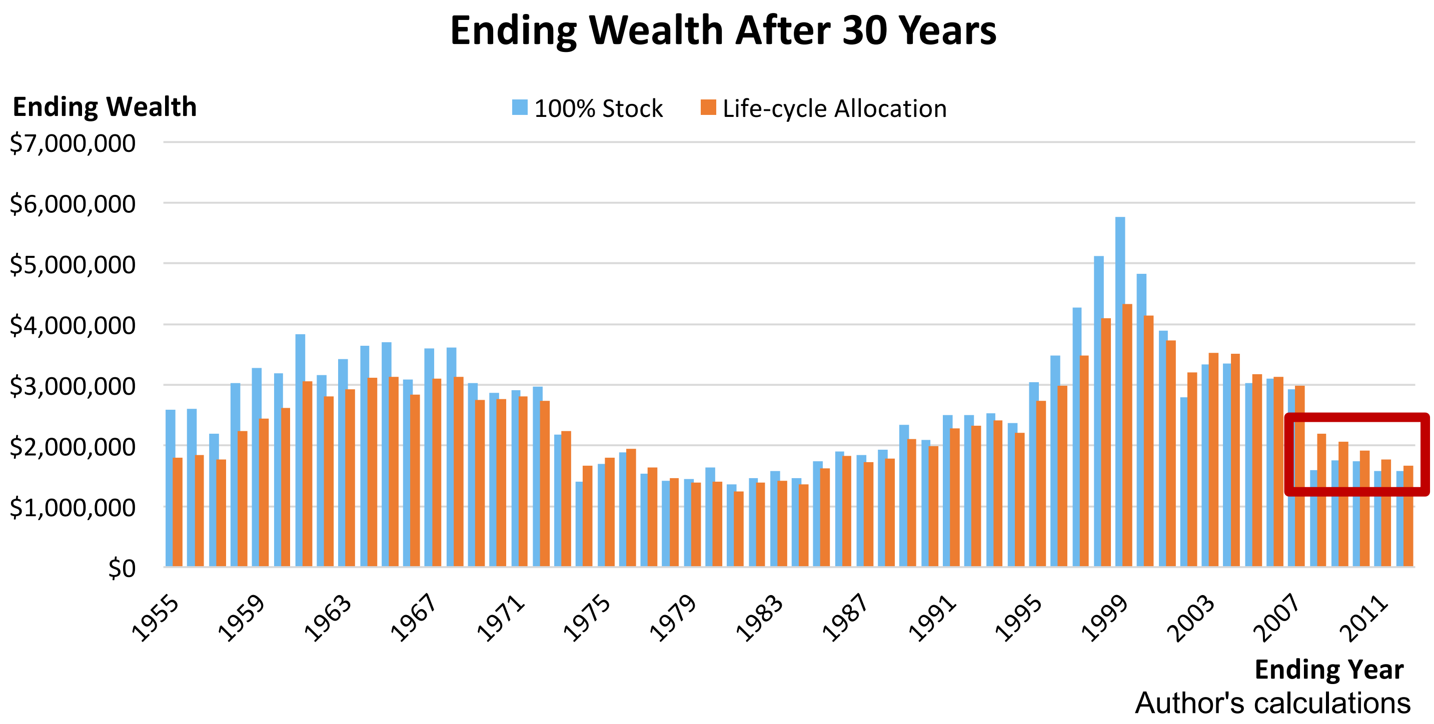


Example life-cycle portfolio



- The returns track each other closely, but the life-cycle strategy yields slightly less risk at the expense of slightly lower average returns.
- The 100% stock portfolio yields average returns of 12.2% with a standard deviation of 1.9%.
- The life-cycle portfolio yields average returns of 11.8% with a standard deviation of 1.7%.

Example life-cycle portfolio



- The life-cycle portfolio misses the post-war and 1990's bull markets.
- But it better withstands the dot-com crash and the Great Recession.
- Regardless of the strategy, ending wealth varies significantly based on economic conditions…

Portfolio Rebalancing

Target Portfolio Allocation

Exposure to different asset classes to achieve a certain risk/reward profile implies a **target allocation** that achieves the desired profile.

Ex. A long-term investor wants a large exposure to stocks for their high expected return, but also wants a small exposure to bonds for diversification benefits. He decides on a **target allocation** of 80% stocks and 20% bonds. He makes total annual contributions of \$10,000 per year, where 80% of the \$10,000 is used to buy stocks and 20% is used to buy bonds. If stocks return 8% per year and bonds return 4% per year, what is his portfolio allocation after 10 years? After 20 years?

Ans. Each year, the investor buys \$8,000 in stock and \$2,000 in bonds. Initially, then, his portfolio consists of 80% stocks and 20% bonds, as desired.

However, after 10 years allocation to stocks grows to 83.4%. After 20 years it further increases to 86.5%.

Stocks

Time Value of Money	
P/Y	1
PV	\$0
PMT	-\$8,000
N	10
I/Y	8%
FV=	\$125,164
N	20
FV=	\$395,383

Bonds

Time Value of Money	
P/Y	1
PV	\$0
PMT	-\$2,000
N	10
I/Y	4%
FV=	\$24,973
N	20
FV=	\$61,938

Portfolio Rebalancing

The above example demonstrates the importance of **portfolio rebalancing**.

- In the absence of any rebalancing, a portfolio's actual allocation may differ from the target allocation if one asset class grows faster than another.
- The effect may be a portfolio with a risk exposure that differs from what is desired. Because of the relatively higher long-term returns to stocks, not actively rebalancing one's portfolio may lead to a high exposure to stocks near or during retirement – just when an investor wants to limit the exposure to financial risk that accompanies stocks.
- In reality, where stock returns are highly volatile and extreme returns year-to-year are not uncommon, the actual allocation can shift even more quickly than suggested by the example above.

Methods for Portfolio Rebalancing

Therefore, a portfolios allocation should be monitored regularly, and portfolio rebalancing may be achieved with the following methods.

- The portfolio may be rebalanced regularly (quarterly, annually, etc.) by selling some of the over-allocated asset class and purchasing the under-allocated asset class. This may trigger capital gains tax liabilities in taxable accounts.
- The portfolio may be rebalanced any time the misallocation of any asset class crosses some threshold. This may also trigger tax liabilities and may require more attention from the investor.
- The portfolio may be rebalanced by altering the allocation of any new contributions such that greater amounts are contributed to under-allocated classes until the target allocation is achieved. This requires more attention from the investor.
- Some funds, such as the life-cycle funds discussed above, are automatically rebalanced by the fund providers.

Rebalancing Bonus

In addition to helping to maintain a desirable risk exposure, diligent portfolio rebalancing can produce a **rebalancing bonus** under certain circumstances.

- If an asset class experiences unusually high returns over a given period, it can quickly disturb the target allocation and require that the asset be sold to restore the target.
- If these high returns are indicative of a bubble in the asset class, selling the asset will reduce the exposure to the resulting collapse in price.
- When the price collapses, the asset will become underallocated and should be repurchased at the new lower price. (And if the market overcorrected in the crash, the stock may be undervalued and a subsequent appreciation may then occur.)
- Therefore, portfolio rebalancing can help investors to buy low and sell high without even thinking!

Interest Rate Risk

Interest rate risk and retirement

The following set of examples considers the negative effect of a drop in interest rates on a retirement portfolio and how a portfolio can be constructed to mitigate this effect.

Ex 1.

An investor is planning to retire at age 65 and would like to live off an income of $50,000 a year. The interest rate is 4%.

If the investor lives solely of interest, how large of a portfolio will he require?

If the investor is willing to withdraw principal, and requires the portfolio to last 30 years, how large of a portfolio will he require?

Interest rate risk and retirement

Ans.

If interest income provides the entire \$50,000, the required principal must satisfy the following relation:

$$\$50{,}000 = 0.04 * P$$

And so the principal must be:

$$P = \frac{\$50{,}000}{0.04} = \$1{,}250{,}000$$

If the investor desires to live off interest, he must accumulate a portfolio of \$1.25 million at retirement…

Interest rate risk and retirement

Ans. (continued)

If the retiree only requires the portfolio to last 30 years and is willing to withdraw some principal each year, he must accumulate only $864,602:

Time Value of Money	
PMT	-$50,000
FV	$0
i	4%
n	30
PV=	$864,602

We've already discussed how amortizing principal in this way comes with the risk that the investor will live longer than the portfolio will last.

There's also the risk that the retiree will not earn interest at the expected rate in retirement. Because a retiree's portfolio should be a conservative mixture of cash and bonds, a retiree's income will be sensitive to the interest rate…

Interest rate risk and retirement

Ex 2.

At age 65, the investor accumulates $865k worth of bank deposits and short-term T-bills earning a 4% interest rate with the plan to withdraw $50,000 a year for 30 years. The interest rate, however, immediately declines from 4% to 1%.

At an interest rate of 1%, how much may the investor withdraw from an $865k portfolio each year over 30 years?

Interest rate risk and retirement

Ans.

Because the interest rate is lower, the portfolio now generates less interest income, so will provide less total income over the investor's retirement.

Given a principal of $865k, an interest rate of 1%, and a period of 30 years, the amount retiree may only withdraw about $33,500 each year:

Time Value of Money	
PV	-$865,000
FV	$0
i	1%
n	30
PV=	$33,517

The lower-than-expected interest rate decreases the retiree's income. Next, we show how this can be guarded against. But first, we must introduce a concept known as **duration**…

Duration

Very early in the lectures we found that **bond prices are inversely related to interest rates**. A bond's **duration** measures this sensitivity.

Specifically, the percent change in a bond's price is (approximately) directly proportional to its duration:

$$\boxed{\%\Delta P \approx -D\,\frac{\Delta r}{1+r}}$$

Where *D* is the bond's duration, r is the initial interest rate, and Δr is the change in the interest rate.

(Note the negative sign because of the inverse relationship between bond prices and interest rates.)

Duration

You will not be asked calculate a bond's duration in this course, but mathematically, the duration of a bond is the weighted average maturity of the bonds cash flows (weighted by the present values of the cash flows).

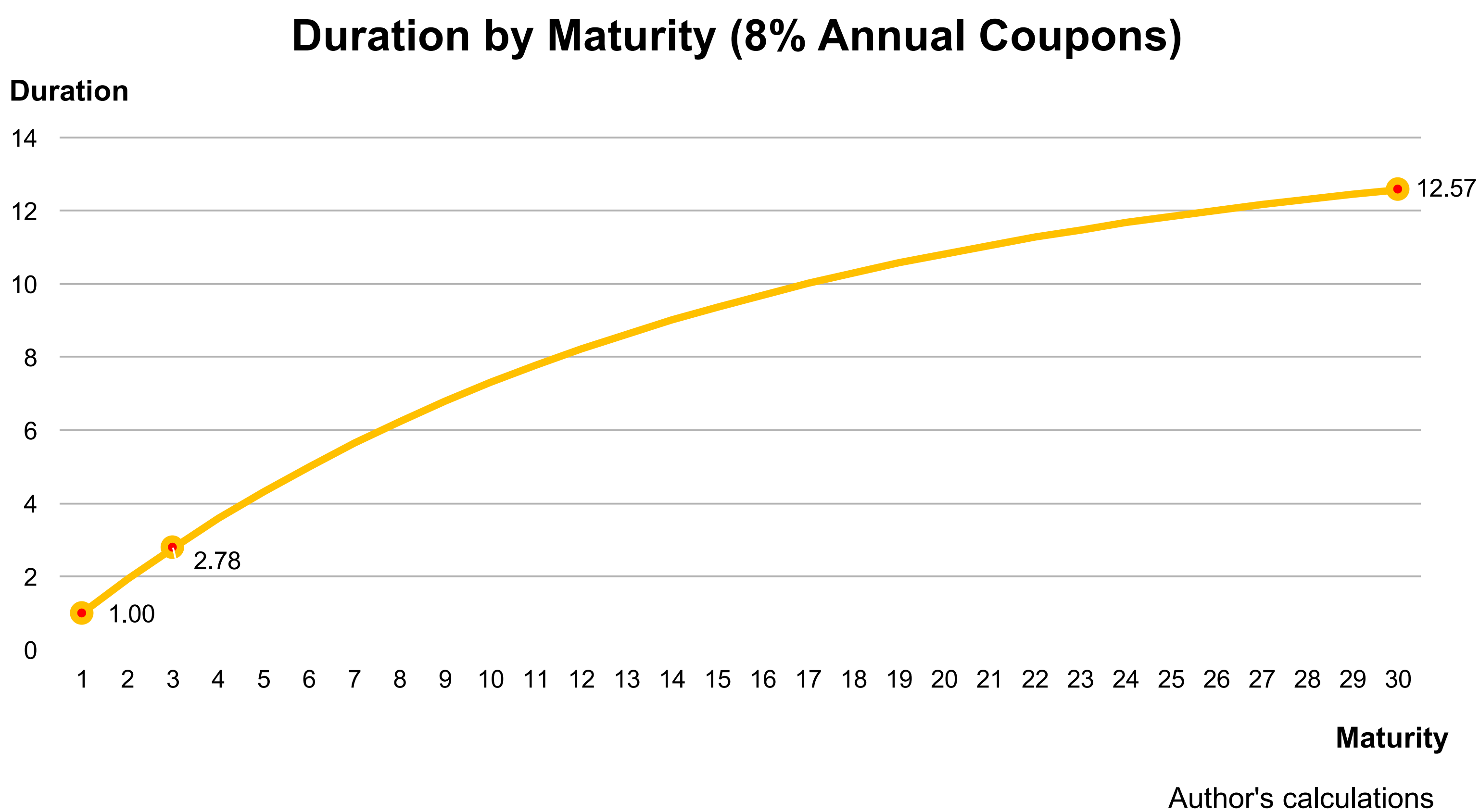
For example, for an annual coupon bond maturing in *T* years and paying coupons of *C* per $100 par, the duration is:

$$D = \frac{1 * \frac{C}{1+r} + 2 * \frac{C}{(1+r)^2} + \cdots + T * \frac{\$100 + C}{(1+r)^T}}{\frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{\$100 + C}{(1+r)^T}}$$

(Note: Duration is sometimes defined slightly differently. This definition is referred to as the **Macaulay Duration**.)

Duration for different maturities

The duration is longer for long-term bonds:



Duration

Ex. The duration of a 3-year 8% annual coupon bond is 2.78 when the interest rate is 7.5%. Calculate the approximate percent change in such a bond's price when the interest rate increases by 50 basis points from 7.5% to 8.0%. Compare this to the actual price change.

Ans. Using the duration formula, we can approximate the percent change in the bond's price as:

$$\%\Delta P \approx -D\frac{\Delta r}{1+r} = -2.78 * \frac{0.005}{1.075} = -1.29\%$$

To find the exact change, first find the bond's price at an interest rate of 7.5%:

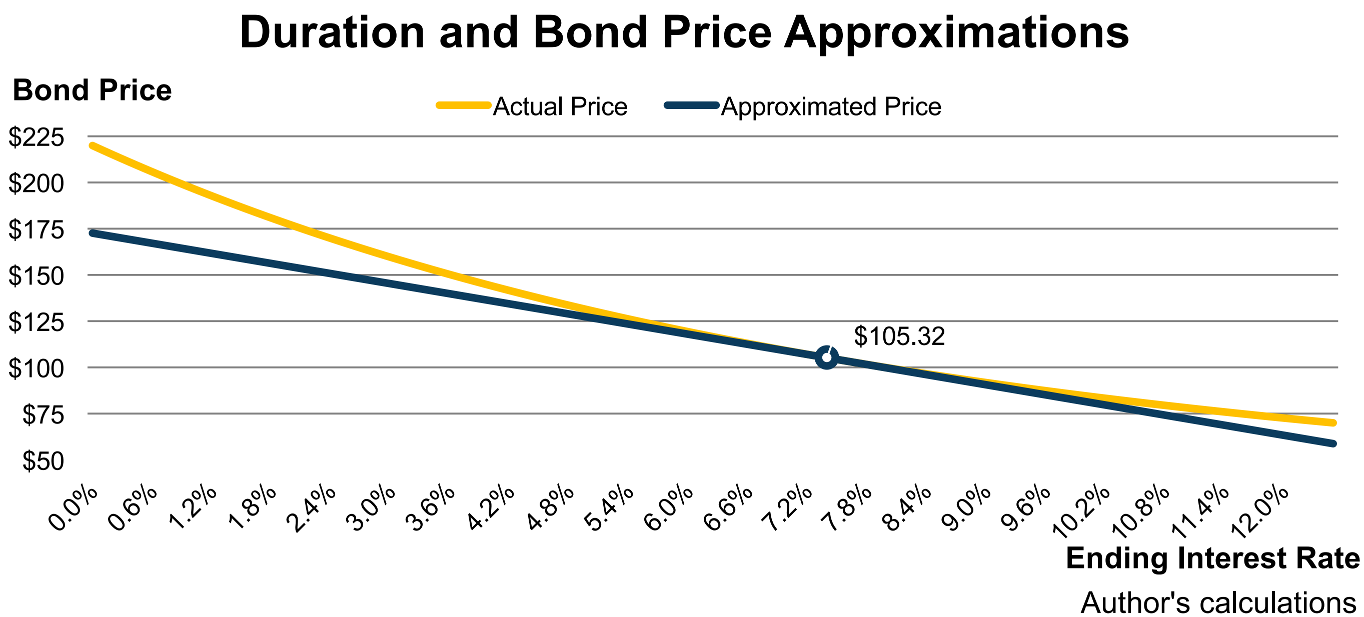
$$P = \frac{\$8}{1.075} + \frac{\$8}{1.075^2} + \frac{\$108}{1.075^3} = \$101.30$$

At an 8% interest rate, the bond will sell at par. The actual change is therefore:

$$\frac{\$100 - \$101.30}{\$101.30} = -1.28\%$$

Duration

The following chart shows the approximated versus actual price of a **15-year** 8% annual coupon bond:



- The bigger the change in r, the worse the approximation.
- The approximation always **underestimates** the price of the bond.
- For athree-year bond, the approximation performs well. For longer-term bonds, the approximation will be less accurate for large interest rate changes.

Duration

The formula for duration need not be memorized, but the following points should be remembered:

- For zero-coupon bond, a bond’s duration is equal to its maturity, because there is only one cash flow.
- The longer a bonds maturity, the longer the duration. This is because the duration is the weighted average maturity of the cash flows, and the largest cash flow occurs at maturity.
- Therefore, longer term bonds are more sensitive to interest rate changes; if interest rates increase, a long-term bond’s price will fall further than a short-term bond’s.
- The smaller the change in the interest rate, the better the approximation. If interest rates change a lot, the approximation will be less accurate.

The duration of a bond **portfolio** is the weighted average durations of the individual bond holdings.

Hedging interest rate risk at retirement

Now, we return to our earlier problem and see that our retiree may hedge against falling interest rates by investing in long-term bonds.

- Earlier, we saw that a retiree with $865k in short-term bonds and bank deposits is exposed to interest rate risk: if the interest rate falls, the retiree's income is reduced.
- Specifically, we considered the case of an investor with his full $865k invested in short-term bonds and bank deposits originally yielding 4%. If the interest rate had remained at this level, he would have been able to withdraw about $50,000 per year in retirement. **However, after interest rates fell from 4% to 1%, we found that he would only be able to withdraw about $33,500 per year.**
- The following examples demonstrate that if the retiree had **instead invested in longer-term bonds, he would be able to maintain a higher retirement income.**

Hedging interest rate risk at retirement

Ex 1. To hedge against the risk that interest rates may fall, a retiree divides his \$865k retirement portfolio **evenly** between bank deposits and annual coupon bonds with maturities of 5, 10, 15, and 20 years. The interest rate on the deposits and coupon rate on the bonds are both 4%.

The duration on the bank deposits is 0. At a 4% interest rate, the durations on the 5, 10, 15, and 20-year bonds are 4.63, 8.44, 11.56, and 14.13, respectively. What is the duration of the portfolio?

When interest rates decline from 4% to 1%, by how much does the portfolio's value change, as estimated using duration?

If the retiree sells his entire portfolio at the new price, and reinvests the proceeds at the new interest rate of 1%, estimate how much the investor may withdraw each year when amortizing the principal over 30 years.

Hedging interest rate risk at retirement

Ans.

The investor places \$865k/5 = \$173k in each of the five investments. Because the duration of a portfolio is the weighted average duration of the holdings in the portfolio, the portfolio's duration can be found to be:

$$D_P = \frac{1}{5} * 0 + \frac{1}{5} * 4.63 + \frac{1}{5} * 8.44 + \frac{1}{5} * 11.56 + \frac{1}{5} * 14.13 = 7.752$$

Given a decline in interest rates from 4% to 1%, the value of the portfolio will then change by:

$$\%\Delta P \approx -D_P * \frac{\Delta r}{1+r} = -7.752 * \frac{-0.03}{1.04} = 22.36\%$$

And the final value of the portfolio may be approximated as:

$$\%\Delta P = \frac{P_F - P_0}{P_0} \rightarrow P_F = P_0 * (1 + \%\Delta P) \approx \$865k * (1 + 0.2236) = \$1{,}058k$$

Hedging interest rate risk at retirement

Ans. (continued)

Therefore, when interest rates fall from 4% to 1%, the portfolio will increase in value to from $865,000 to approximately $1,058,000!

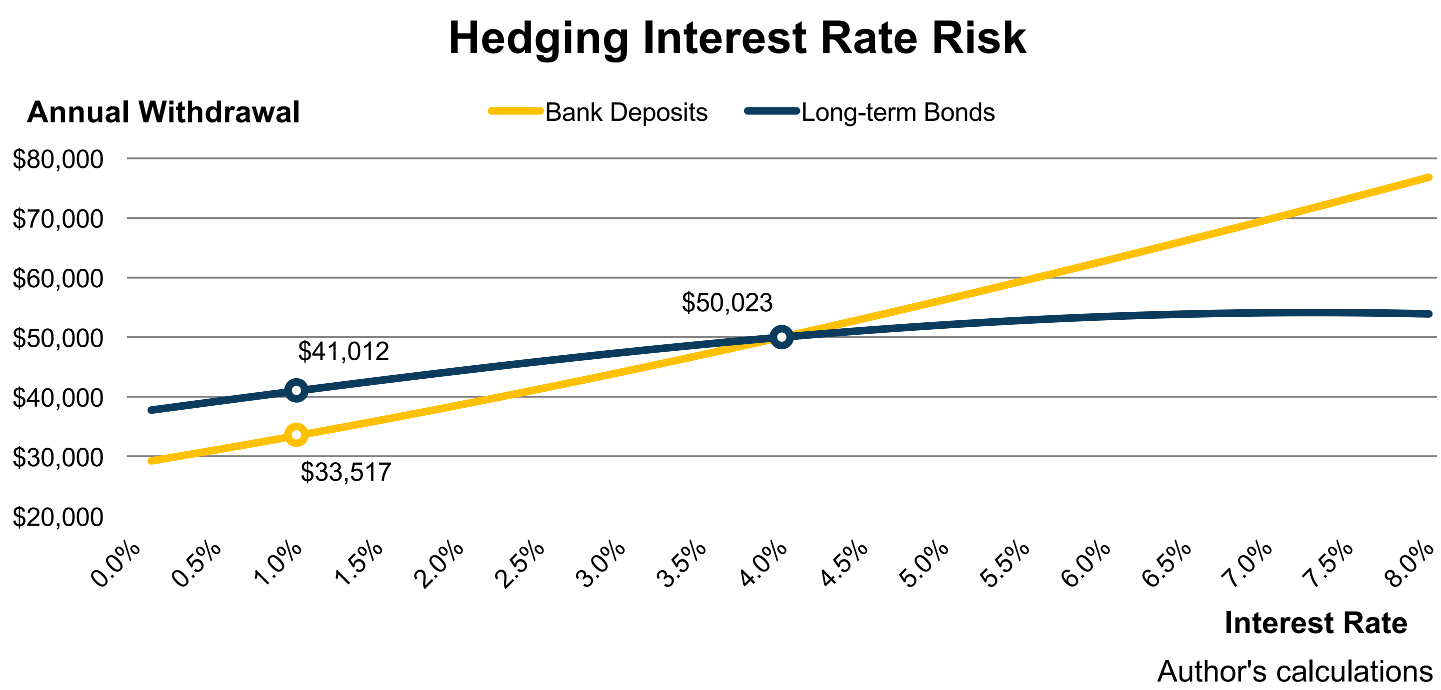
The retiree had planned to withdraw $50k per year. If everything was in short-term bonds and bank deposits (with a duration of zero), the change in interest rates would reduce his withdrawals to $33.5k. If he hedges the interest rate risk by investing in long-term bonds, he may still withdraw about $41k each year:

Time Value of Money	
PV	-$1,058,000
FV	$0
i	1%
n	30
PV=	$40,996

(Remember that duration *underestimates the price change*. In fact, the portfolio would now be worth about $1,105k and sustain annual withdrawals of $42.8k.)

Hedging interest rate risk at retirement

The following chart shows our retiree's allowable annual withdrawal at various interest rates:



Note that by hedging against interest rate changes, the retiree also sacrifices the potential benefit from increased interest rates...

Hedging interest rate risk at retirement

A retiree may protect against interest rate changes by holding a portfolio composed of long-term bonds.

- A retiree may **lock-in** an interest rate by purchasing bonds that promise to pay that coupon rate over the retirement horizon.
- The more closely a retiree can **match the bond's cash flows** to his or her income needs, the better. For example, if the retiree above purchased zero coupon bonds maturing each year for the next 30 years with face values of $50,000, he would face no interest rate risk.
- To the extent the cash flows don't match *exactly*, there will be some interest rate risk because the retiree will have to either reinvest bond proceeds at a different rate or sell some of the bonds at a different price.
- By hedging against interest rate risk, however, a retiree also **hedges against increasing interest rates** and loses any potential increases in income. This is especially **bad when inflation increases** – the fixed nominal income will decrease in real terms.

Longevity Risk and Annuities

Longevity risk

To properly plan for retirement, it's necessary to estimate how long a retirement fund must last.

- Because life expectancy is uncertain, there is a risk that retirees might outlive their retirement funds.
- This risk is known as **longevity risk**.
- One way to mitigate this risk is to **save enough to last longer than you can reasonably expect to live**. That way, if you do actually live longer than expected, you will still have sufficient funds.
- Another way to protect against this risk is to purchase a product known as a **life annuity…**

Life annuities

Annuities are financial contracts that offer a stream of contractual payments in exchange for a lump sum today.

- A **term annuity** provides the annuitant a stream of cash flows for a set and pre-specified number of years. A **life annuity** provides cash flows for as long as the annuitant lives.
- An **immediate annuity** begins making payments today in exchange for a lump sum. In a **deferred annuity**, the annuitant builds up the value of the annuity over time before selecting a time to receive payments.
- **Fixed** annuities offer a fixed stream of payments over time, while the payments on a **variable** annuity change based on some underlying index.

Life annuities

A retiree may guard against longevity and financial risk by **annuitizing** a portion of his or her retirement portfolio.

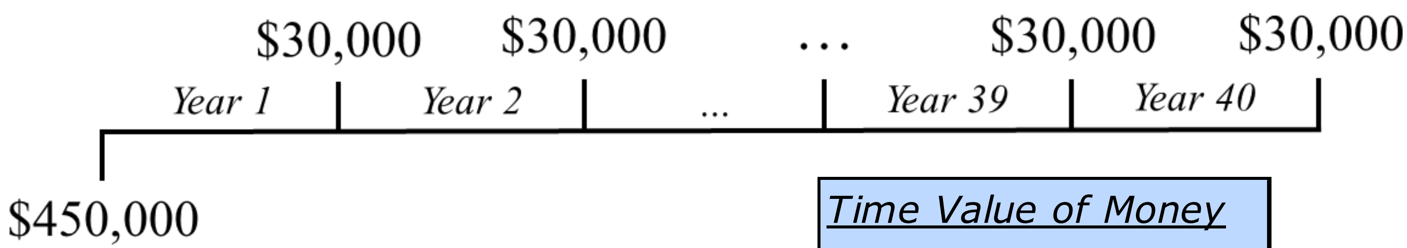
Ex. A 65 year-old male in the District of Columbia is set to retire and determines that he will require at least $30,000 a year to meet his basic needs during retirement. He currently has $800,000 in retirement funds and finds that he can purchase an immediate-, fixed-, life-annuity for $450,000 that will provide $30,000 per year until he dies.

He is considering whether to purchase the annuity or not. Benchmark the annuity by calculating the implicit return on the annuity, assuming the retiree lives until age 105.

Life annuities

Ans.

Over 40 years, such an annuity will offer the following payments:



The implied return on such a contract is:

Time Value of Money

PV	-$450,000
FV	$0
PMT	$30,000
n	40
i=	6.0%

The retiree would like a guaranteed income of $30,000 for at least 40 years to protect again longevity risk. Given that he is not certain he can consistently beat a 6% return by investing in stocks and bonds, he elects to purchase the annuity for $450,000.

Life annuities

The return on such an annuity for different life expectancies and the probability of living to each age are:

Age	Probability	IRR on Annuity
70	92.1%	-28.2%
75	80.9%	-6.7%
80	65.7%	0.0%
85	46.6%	2.9%
90	25.8%	4.4%
95	9.3%	5.2%
100	1.9%	5.7%
105	0.2%	6.0%
110	0.0%	6.2%

Author's calculations

Life annuities

Annuities should be though of as an insurance contract against longevity and financial risk:

- The returns to an annuity will be low if the annuitant does not live long, or even to the average life expectancy of 85.
- But the table above demonstrates that a large portion of the retirees will live to 95, and some will even live to 105.
- At these longevities, the annuity provides a higher implicit return that might not be comfortably recreated by the conservative portfolios of retirees.
- Annuities, however, do not protect against **inflation** and are also subject to **default risk** (AIG, for example, underwrote a lot of annuities).

Social Security

Returns to Social Security

In the second lecture, we analyzed the returns to Social Security. For a hypothetical (but realistic) scenario we calculated the implied returns to Social Security at different life expectancies for an average worker:

Age of Death	Total Benefit	Return
75	$220,000	-0.48%
85	$440,000	1.87%
95	$660,000	2.77%
105	$880,000	3.22%

We noted that these returns are lower than the long-run returns offered by other financial products in the market…

Risk and Social Security

But while the returns may be lower, Social Security offers protections that other products cannot match:

- Social Security is not subject to **financial risk**. While stocks may decline in value, Social Security payments do not.
- Unlike bonds, Social Security payments are free of **default risk**. The payments are guaranteed by the U.S. government.
- Social Security is not affected by **interest rate risk**. If interest rates decrease, Social Security payments will not.
- Social Security payments are adjusted for **inflation**.
- Social Security provides insurance against **longevity risk**. Social Security payments are made until a beneficiary dies, and so the payments cannot be outlived.
- However, Social Security is subject to **political risk**. If Congress decides to reform Social Security, the payments may differ from what was expected…

Today we learned…

- ✓ Leverage and risk
- ✓ Interest and leverage
- ✓ Leverage and bankruptcy
- ✓ Life-cycle investing
- ✓ Portfolio rebalancing
- ✓ Interest rate risk
- ✓ Longevity risk and annuities
- ✓ Social Security